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Ground state for the Schrödinger operator with the weighted Hardy potential

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Abstract

We establish the existence of ground states on $\mathbb{R}^N$ for the Laplace operator involving the Hardy type potential. This gives rise to the existence of the principal eigenfunctions for the Laplace operator involving weighted Hardy potentials. We also obtain a higher integrability property for the principal eigenfunction. This is used to examine the behaviour of the principal eigenfunction around 0.

1 Introduction

In this paper we investigate the existence of ground states of the Schrödinger operator associated with the quadratic form

$$Q_V(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \Lambda_V V(x) u^2\right) dx,\ u \in C_\circ^\infty(\mathbb{R}^N),\ N \geq 3, \tag{1.1}$$

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where V belongs to the Lorentz space $L^{\frac{N}{2},\infty}(\mathbb{R}^N)$ and Λ_V is the largest constant (whenever exists) for which the form Q_V is nonnegative. This assumption implies that the potential term $\int_{\mathbb{R}^N} V(x)u^2\,dx$ is continuous in $D^{1,2}(\mathbb{R}^N)$, where $D^{1,2}(\mathbb{R}^N)$ is the Sobolev space obtained as the completion of $C_\circ^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

We are mainly interested in the case of the Hardy type potential $V(x) = \frac{m(x)}{|x|^2}$ with $m \in L^\infty(\mathbb{R}^N)$. Assuming that V is positive on a set of positive measure, the constant Λ_V is given by the variational problem

$$\Lambda_V = \inf_{u\in D^{1,2}(\mathbb{R}^N),\, \int_{\mathbb{R}^N} Vu^2\,dx=1} \int_{\mathbb{R}^N} |\nabla u|^2\,dx \tag{1.2}$$

and the continuity of $\int_{\mathbb{R}^N} V(x)u^2\,dx$ implies that $\Lambda_V > 0$. If problem (1.2) has a minimizer u, then it satisfies the equation

$$-\Delta u - \Lambda_V V(x)u = 0. \tag{1.3}$$

A solution of (1.3) is understood in the weak sense

$$\int_{\mathbb{R}^N} \nabla u \nabla\phi\,dx = \Lambda_V \int_{\mathbb{R}^N} V(x)u\phi\,dx, \tag{1.4}$$

for every $\phi \in D^{1,2}(\mathbb{R}^N)$.

Since $|u|$ is also a minimizer for Λ_V, we may assume that $u \geq 0$ a.e. on $\mathbb{R}^N$. In particular, when $V(x) = \frac{m(x)}{|x|^2}$ with $m \in L^\infty(\mathbb{R}^N)$, then $u > 0$ on $\mathbb{R}^N$ by the Harnack inequality [14]. If the potential term is weakly continuous in $D^{1,2}(\mathbb{R}^N)$, for example, when $V(x) = \frac{m(x)}{|x|^2}$ with $m \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} m(x) = \lim_{x\to 0} m(x) = 0$, then there exists a minimizer for Λ_V. We will call the minimizer of (1.2) a *ground state of finite energy*. In general, (1.2) may not have a minimizer. This is the case for the Hardy potential $V(x) = \frac{1}{|x|^2}$ with the corresponding optimal constant $\Lambda_V = \Lambda_N = \left(\frac{N-2}{2}\right)^2$. In fact, the ground state of finite energy is a particular case of the generalized ground state, defined as follows (see [24], [26] and [27]).

Definition 1.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set, and let Q_V be as in (1.1). A sequence of nonnegative functions $v_k \in C_\circ^\infty(\Omega)$ is said to be a null-sequence for the functional Q_V if $Q_V(v_k) \to 0$, as $k \to \infty$, and there exists a nonnegative function $\psi \in C_\circ^\infty(\Omega)$ such that $\int_\Omega \psi v_k\,dx = 1$ for each k.*

Let us recall that the capacity of a compact set E relative to an open set $\Omega \subset \mathbb{R}^N$, with $E \subset \Omega$, is given by

$$\text{cap }(E,\Omega) = \inf\{\int_\Omega |\nabla u|^2\,dx;\ u \in C_\circ^\infty(\Omega), \text{ with } u(x) \geq 1 \text{ on } E\}.$$

In the case $\Omega = \mathbb{R}^N$ we use notation cap (E) (see [23]).

We can now formulate the following "ground state alternative" (see [26], [27]).

Theorem 1.2 *Let V be a measurable function bounded on every compact subset of $\Omega = \mathbb{R}^N - Z$, where Z is a closed set of capacity zero, and assume that $Q_V(u) \geq 0$ for all $u \in C_\circ^\infty(\Omega)$. Then, if Q_V admits a null sequence v_k, then the sequence v_k converges weakly in $H^1_{\text{loc}}(\mathbb{R}^N)$ to a unique (up to a multiplicative constant) positive solution of (1.3).*

This theorem gives rise to the definition of the generalized ground state.

Definition 1.3 *A unique positive solution v of (1.3) is called a generalized ground state of the functional Q_V, if the functional admits a null sequence weakly convergent to v.*

If $V(x) = \frac{1}{|x|^2}$, the functional Q_V has a ground state $v(x) = |x|^{\frac{2-N}{2}}$ of infinite $D^{1,2}$ norm, while (1.2) has no minimizer in $D^{1,2}(\mathbb{R}^N)$.

It is important to note that the functional Q_V with the optimal constant Λ_V does not necessarily have a ground state. We quote the following statement from [27].

Theorem 1.4 *Let V be a measurable function bounded on every compact subset of $\Omega = \mathbb{R}^N - Z$, where Z is a closed set of capacity zero, and assume that $Q_V(u) \geq 0$ for all $u \in C_\circ^\infty(\Omega)$. Then either Q_V admits a null sequence, or there exists a function W, positive and continuous on Ω, such that*

$$Q_V(u) \geq \int_{\mathbb{R}^N} W(x) u^2 \, dx. \tag{1.5}$$

For example, let m be a continuous function on $\mathbb{R}^N - \{0\}$ such that $m(x) = \frac{1}{|x|^2}$ for $0 < |x| \leq 1$, $m(x) \in [\frac{1}{2}, 1]$ for $|x| \in (1, 2)$ and $m(x) = \frac{1}{2|x|^2}$ for $|x| \geq 2$. Then $\Lambda_V = \left(\frac{N-2}{2}\right)^2$ and the functional Q_V does not admit a null sequence. From Theorem 1.4 follows that Q_V satisfies (1.5) with some function W positive on $\mathbb{R}^N - \{0\}$.

Obviously, ground states of finite $D^{1,2}$ norm are principal eigenfunctions of (1.3). There is a quite extensive literature on principal eigenfunctions with indefinite weight functions for elliptic operators on $\mathbb{R}^N$, or on unbounded domains of $\mathbb{R}^N$, with the Dirichlet boundary conditions. We mention papers [2], [6], [7], [15], [19], [24], [29], [30], [31], where the existence of principal eigenfunctions has been established under various assumptions on weight functions. These conditions require that a potential belongs to some Lebesgue space, for example $L^p(\mathbb{R}^N)$ with $p > \frac{N}{2}$. These results have been recently greatly improved in papers [3] and [33], where potentials from the Lorentz spaces have been considered. To describe the results from [3] and [33] we recall the definition of the Lorentz space [5], [18], [21].

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We define the distribution function α_f and a nonincreasing rearrangement f^* of f in the following way

$$\alpha_f(s) = |\{x \in \mathbb{R}^N;\ |f(x)| > s\} \quad \text{and} \quad f^*(t) = \inf\{s > 0;\ \alpha_f(s) \leq t\}.$$

We now set

$$\|f\|^*_{(p,q)} = \begin{cases} \left(\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 1 \leq p, q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } 1 \leq p \leq \infty, q = \infty. \end{cases}$$

The Lorentz space $L^{p,q}(\mathbb{R}^N)$ is defined by

$$L^{p,q}(\mathbb{R}^N) = \{f \in L^1_{\text{loc}}(\mathbb{R}^N);\ \|f\|^*_{(p,q)} < \infty\}.$$

The functional $\|f\|^*_{(p,q)}$ is only a quasi-norm. To obtain a norm we replace f by $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, dx$ in the definition of $\|f\|^*_{(p,q)}$, that is, the norm is given by

$$\|f\|_{(p,q)} = \begin{cases} \left(\int_0^\infty [t^{\frac{1}{p}} f^{**}(t)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 1 \leq p, q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & \text{if } 1 \leq p \leq \infty, q = \infty. \end{cases}$$

$L^{p,q}(\mathbb{R}^N)$ equipped with the norm $\|f\|_{(p,q)}$ is a Banach space.

In paper [33] the existence of principal eigenfunctions has been established for weights belonging to $\bigcup_{1\leq q<\infty} L^{\frac{N}{2},q}(\mathbb{R}^N)$. This was extended in [3] to a larger class of weights $\mathcal{F}_{\frac{N}{2}}$ obtained as the completion of $C^\infty_\circ(\mathbb{R}^N)$ in norm $\|\cdot\|_{\frac{N}{2},\infty}$.

However, these conditions do not cover the singular weight functions considered in this paper. By contrast, in our approach we give an exact upper bound for the principal eigenvalue which allows us to prove the existence of the principal eigenfunction. We point out that if $V \in L^{\frac{N}{2},\infty}(\mathbb{R}^N)$, then the functional $\int_{\mathbb{R}^N} V(x)u^2\, dx$ is continuous on $D^{1,2}(\mathbb{R}^N)$, but not necessarily weakly continuous.

The paper is organized as follows. In Section 2 we prove the existence of minimizers with finite norm $D^{1,2}(\mathbb{R}^N)$ and also with infinite norm $D^{1,2}(\mathbb{R}^N)$. In Section 3 we discuss perturbation of a given quadratic form $Q_{V_\circ}$ with $V_\circ \in L^{\frac{N}{2},\infty}(\mathbb{R}^N)$. We show that if $Q_{V_\circ}$ has ground state, then this property is stable under small perturbations of $V_\circ$. This is not true if $Q_{V_\circ}$ does not have a ground state; rather it is stable under larger perturbation of $V_\circ$. The final Section is devoted to a higher integrability property of minimizers of $Q_{V_\circ}$ in the case where $V_\circ(x) = \frac{m(x)}{|x|^2}$ with $m \in L^\infty(\mathbb{R}^N)$. We also examine the behaviour of the principal eigenfunction around 0.

Throughout this paper, in a given Banach space we denote strong convergence by " $\to$ " and weak convergence by " $\rightharpoonup$ ". The norms in the Lebesgue space $L^p(\Omega)$, $1 \leq p \leq \infty$, are denoted by $\|u\|_p$.

2 Existence of minimizers

We consider the Hardy type potential $V(x) = \frac{m(x)}{|x|^2}$ with $m \in L^\infty(\mathbb{R}^N)$. In Theorem 2.2 we formulate conditions on m guaranteeing the existence of a principal eigenfunction. Let $\gamma_+ > 1$ and $\gamma_- > 1$. In our approach to problem (1.2) the following two limits play an important role: it is assumed that the following limits exist a.e.

$$m_+(x) = \lim_{j\in\mathbb{N}, j\to\infty} m(\gamma_+^j x) \tag{2.1}$$

and

$$m_-(x) = \lim_{j\in\mathbb{N}, j\to\infty} m(\gamma_-^{-j} x). \tag{2.2}$$

Both functions $m_\pm$ satisfy $m_\pm(\gamma_\pm x) = m_\pm(x)$, that is, $m_\pm$ are homogeneous of degree 0. We now define the following infima:

$$\Lambda_m = \inf_{u\in D^{1,2}(\mathbb{R}^N)-\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} u^2\,dx}, \tag{2.3}$$

(we use the notation Λ_m instead of Λ_V) and

$$\Lambda_\pm = \inf_{u\in D^{1,2}(\mathbb{R}^N)-\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N} \frac{m_\pm(x)}{|x|^2} u^2\,dx}. \tag{2.4}$$

Lemma 2.1 *The following holds true*

$$\Lambda_m \le \min(\Lambda_+, \Lambda_-). \tag{2.5}$$

Proof Let $u \in D^{1,2}(\mathbb{R}^N) - \{0\}$. Testing Λ_m with $\gamma_+^{-\frac{N-2}{2}} u(\gamma_+^{-j} x)$ gives

$$\Lambda_m \le \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N} \frac{m(\gamma_+^j x)}{|x|^2} u^2\,dx}.$$

Letting $j \to \infty$ and using the Lebesgue dominated convergence theorem, we obtain

$$\Lambda_m \le \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N} \frac{m_+(x)}{|x|^2} u^2\,dx}.$$

The inequality $\Lambda_m \le \Lambda_+$ follows. The proof of the inequality $\Lambda_m \le \Lambda_-$ is similar. □

In the case when the inequality (2.4) is strict problem (2.2) has a minimizer.

Theorem 2.2 *Assume that the convergence in (2.1) is uniform on sets $\{x \in \mathbb{R}^N;\ |x| \ge R\}$ for every $R > 0$ and that the convergence in (2.2) is uniform on sets $\{x \in \mathbb{R}^N;\ |x| \le \rho\}$ for every $\rho > 0$. If $\Lambda_m < \min(\Lambda_+, \Lambda_+)$, then problem (2.3) has a minimizer.*

Proof Let $\{u_k\} \subset D^{1,2}(\mathbb{R}^N)$ be a minimizing sequence for Λ_m, that is,

$$\int_{\mathbb{R}^N} |\nabla u_k|^2\,dx \to \Lambda_m \quad\text{and}\quad \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} u_k^2\,dx = 1.$$

We can assume, up to a subsequence, that $u_k \rightharpoonup w$ in $D^{1,2}(\mathbb{R}^N)$, $L^2(\mathbb{R}^N, \frac{dx}{|x|^2})$ and $u_k \to w$ in ${L^2}_{\text{loc}}(\mathbb{R}^N)$ for some $w \in D^{1,2}(\mathbb{R}^N)$. Let $v_k = u_k - w$. We then have

$$1 = \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} u_k^2\,dx = \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} w^2\,dx + \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} v_k^2\,dx + o(1) \tag{2.6}$$

and

$$\Lambda_m = \int_{\mathbb{R}^N} |\nabla u_k|^2\,dx + o(1) = \int_{\mathbb{R}^N} |\nabla w|^2\,dx + \int_{\mathbb{R}^N} |\nabla v_k|^2\,dx + o(1). \tag{2.7}$$

We define a radial function $\chi_+^j \in C^1(\mathbb{R}^N)$ such that $0 \le \chi_+^j(x) \le 1$, $\chi_+^j(x) = 0$ for $|x| \le \gamma_-^{-2j}$ and $\chi_+^j(x) = 1$ for $|x| > \gamma_+^{2j}$. Let $\chi_-^j(x) = 1 - \chi_+^j(x)$. In what follows we use $o^{(j)}_{k\to\infty}(1)$ to denote a quantity such that for each $j \in \mathbb{N}$, $o^{(j)}_{k\to\infty}(1) \to 0$ as $k \to \infty$. Thus

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} v_k^2\,dx &= \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} \big(v_k \chi_-^j\big)^2\,dx + \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} \big(v_k \chi_+^j\big)^2\,dx + o^{(j)}_{k\to\infty}(1) \\ &= \int_{\mathbb{R}^N} \frac{m(\gamma_-^{-j}x)}{|x|^2} \big(v_k^-\big)^2\,dx + \int_{\mathbb{R}^N} \frac{m(\gamma_+^{j}x)}{|x|^2} \big(v_k^+\big)^2\,dx + o^{(j)}_{k\to\infty}(1), \end{aligned} \tag{2.8}$$

where

$$v_k^-(x) = \gamma_-^{-\frac{N-2}{2}j} v_k\big(\gamma_-^{-j}x\big)\chi_-\big(\gamma_-^{-j}x\big)$$

and

$$v_k^+(x) = \gamma_+^{\frac{N-2}{2}j} v_k\big(\gamma_+^{j}x\big)\chi_+\big(\gamma_+^{j}x\big).$$

We now estimate the integrals involving v_k^- and v_k^+. We have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \frac{m\big(\gamma_-^{-j}x\big)}{|x|^2} (v_k^-)^2\,dx \right. &- \left. \int_{\mathbb{R}^N} \frac{m_-(x)}{|x|^2} (v_k^-)^2\,dx \right| \le \left| \int_{|x|<\gamma_-^{-j}} \frac{m\big(\gamma_-^{-j}x\big) - m_-(x)}{|x|^2} (v_k^-)^2\,dx \right| \\ &+ \left| \int_{\gamma_-^{-j}<|x|<\gamma_-^{j}\gamma_+^{2j}} \frac{m(\gamma_-^{-j}x) - m_-(x)}{|x|^2} (v_k^-)^2\,dx \right| = J_1 + J_2. \end{aligned}$$

By the uniform convergence of $m\big(\gamma_-^{-j}x\big)$ to $m_-(x)$ we see that $J_1 \le \epsilon$ for j sufficiently large uniformly in k. For J_2 we have

$$J_2 \le 2\|m\|_\infty \int_{\gamma_-^{-2j}<|x|<\gamma_+^{2j}} \frac{v_k^2}{|x|^2}\,dx.$$

It is clear that J_2 is a quantity of type $o^{(j)}_{k\to\infty}(1)$. Therefore, we have

$$\left| \int_{\mathbb{R}^N} \frac{m\big(\gamma_-^{-j}x\big)}{|x|^2} (v_k^-)^2\,dx - \int_{\mathbb{R}^N} \frac{m_-(x)}{|x|^2} (v_k^-)^2\,dx \right| \le \epsilon + o_j(1). \tag{2.9}$$

In a similar way we obtain

$$\left| \int_{\mathbb{R}^N} \frac{m(\gamma_+^j x)}{|x|^2} (v_k^+)^2 \, dx - \int_{\mathbb{R}^N} \frac{m_+(x)}{|x|^2} (v_k^+)^2 \, dx \right| \leq \o_{k\to\infty}^{(j)}(1). \tag{2.10}$$

for j sufficiently large. We now fix $j \in \mathbb{N}$ so that (2.9) and (2.10) hold. Consequently, we have

$$1 \leq \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} w^2 \, dx + \int_{\mathbb{R}^N} \frac{m_-(x)}{|x|^2} (v_k^-)^2 \, dx + \int_{\mathbb{R}^N} \frac{m_+(x)}{|x|^2} (v_k^+)^2 \, dx + 2\epsilon + o_{k\to\infty}^{(j)}(1). \tag{2.11}$$

We now estimate $\int_{\mathbb{R}^N} |\nabla v_k|^2 \, dx$ in the following way

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla v_k|^2 \, dx &= \int_{\mathbb{R}^N} |\nabla (v_k \chi_-^j + v_k \chi_+^j)|^2 \, dx \\
&= \int_{\mathbb{R}^N} |\nabla (v_k \chi_-^j)|^2 \, dx + \int_{\mathbb{R}^N} |\nabla (v_k \chi_+^j)|^2 \, dx \\
&+ 2 \int_{\mathbb{R}^N} \nabla (v_k \chi_-^j) \nabla (v_k \chi_+^j) \, dx \\
&= \int_{\mathbb{R}^N} |\nabla v_k^-|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_k^+|^2 \, dx + 2 \int_{\mathbb{R}^N} |\nabla v_k|^2 \chi_-^j \chi_+^j \, dx \\
&+ 2 \int_{\mathbb{R}^N} v_k \nabla v_k \nabla \chi_-^j \chi_+^j \, dx + 2 \int_{\mathbb{R}^N} v_k \nabla v_k \chi_-^j \nabla \chi_+^j \, dx \\
&+ 2 \int_{\mathbb{R}^N} v_k^2 \nabla \chi_-^j \nabla \chi_+^j \, dx \\
&\geq \int_{\mathbb{R}^N} |\nabla v_k^-|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_k^+|^2 \, dx + 2 \int_{\mathbb{R}^N} v_k \nabla v_k \nabla \chi_-^j \chi_+^j \, dx \\
&+ 2 \int_{\mathbb{R}^N} v_k \nabla v_k \chi_-^j \nabla \chi_+^j \, dx \\
&+ 2 \int_{\mathbb{R}^N} v_k^2 \nabla \chi_-^j \nabla \chi_+^j \, dx.
\end{aligned}$$

Since $v_k \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ we obtain the following estimate

$$\int_{\mathbb{R}^N} |\nabla v_k|^2 \, dx \geq \int_{\mathbb{R}^N} |\nabla v_k^-|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_k^+|^2 \, dx + o_{k\to\infty}^{(j)}(1).$$

This, combined with (2.6), gives the following estimate

$$\begin{aligned}
\Lambda_m &\geq \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_k^-|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_k^+|^2 \, dx + o_j(1) \\
&\geq \Lambda_m \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} w^2 \, dx + \Lambda_- \int_{\mathbb{R}^N} \frac{m_-(x)}{|x|^2} (v_k^-)^2 \, dx \\
&+ \Lambda_+ \int_{\mathbb{R}^N} \frac{m_+(x)}{|x|^2} (v_k^+)^2 \, dx + o_{k\to\infty}^{(j)}(1).
\end{aligned} \tag{2.12}$$

Let $\Lambda_* = \min\big(\Lambda_-, \Lambda_+\big)$. We deduce from (2.11) and (2.12) that

$$\big(\Lambda_* - \Lambda_m\big)\left(\int_{\mathbb{R}^N} \frac{m_-(x)}{|x|^2}(v_k^-)^2\,dx + \frac{m_+(x)}{|x|^2}(v_k^+)^2\,dx\right) \le 2\epsilon\Lambda_m + o^{(j)}_{k\to\infty}(1).$$

Letting $k \to \infty$ we obtain

$$\limsup_{k\to\infty}\left(\int_{\mathbb{R}^N} \frac{m_-(x)}{|x|^2}(v_k^-)^2\,dx + \frac{m_+(x)}{|x|^2}(v_k^+)^2\,dx\right) \le \frac{2\epsilon\Lambda_m}{\big(\Lambda_* - \Lambda_m\big)}.$$

It then follows from (2.11) that

$$1 \le \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2}w^2\,dx + \frac{2\epsilon\Lambda_m}{\big(\Lambda_* - \Lambda_m\big)}.$$

Since $\epsilon > 0$ is arbitrary we get $\int_{\mathbb{R}^N} \frac{m(x)}{|x|^2}w^2\,dx = 1$ and the result follows. $\square$

In what follows, we use denote by $m(\infty) = \lim_{|x|\to\infty} m(x)$, assuming that this limit exists. As a direct consequence of Theorem 2.2 we obtain the following result.

Theorem 2.3 *Let $m \in L^\infty(\mathbb{R}^N)$ and assume that m is continuous at 0. Further, suppose that $m(\infty) > 0$ and $m(0) > 0$. If $\Lambda_m < \Lambda_N \min\big(\frac{1}{m(\infty)}, \frac{1}{m(0)}\big)$, then there exists a minimizer for Λ_m.*

Remark 2.4 *Λ_m has a minimizer also in the following cases, corresponding formally to Λ_+ or Λ_- taking the value $+\infty$.*

(i) Let $m(0) = 0$ and $m(\infty) > 0$. If $\Lambda_m < \frac{\Lambda_N}{m(\infty)}$, then a minimizer for $\Lambda_1(m)$ exists.

(ii)Let $m(0) > 0$ and $m(\infty) = 0$. If $\Lambda_m < \frac{\Lambda_N}{m(0)}$, then a minimizer for $\Lambda_1(m)$ exists.

(iii) If $m(0) = m(\infty) = 0$, $m(x) \ge 0$ and $\not\equiv 0$ on $\mathbb{R}^N$, then Λ_m has a minimizer.

We point out that Theorem 2.3 and the results described in Remark 2.4 can be deduced from Theorem 1.2 in [32]. Unlike in paper [32], to obtain Theorem 2.3 we avoided the use of the concentration - compactness principle.

We now give examples of weight functions m satisfying conditions of Theorems 2.2 and 2.3. In general, functions satisfying this condition have large local maxima.

Example 2.5 *Let*

$$m_A(x) = \begin{cases} m_1(x) & \textit{for } 0 < |x| < 1, \\ Am_2(x) & \textit{for } 1 \le |x| \le 2, \\ m_3(x) & \textit{for } 2 < |x|, \end{cases}$$

where $A > 0$ is a constant to be chosen later and $m_1 : \overline{B(0,1) - \{0\}} \to [0,\infty)$, $m_2 :$ $(1 \le |x| \le 2) \to [0,\infty)$ and $m_3 : \mathbb{R}^N \setminus B(0,2) \to [0,\infty)$ are continuous bounded functions

satisfying the following conditions: $m_1(x) = 0$ *for* $|x| = 1$, $m_2(x) = 0$ *for* $|x| = 1$, $m_2(x) = 0$ *for* $|x| = 2$, $m_2(x) > 0$ *for* $1 < |x| < 2$, $m_3(x) = 0$ *for* $|x| = 2$. *Further we assume that*

$$m_3(x) = \frac{a + |x_1||x_2| + \ldots + |x_{N-1}||x_N|}{b + |x|^2}$$

for $|x| \geq R > 2$, *where* $a > 0$, $b > 0$ *and* R *constants. A function* $m_1(x)$ *for small* $\delta > 0$ *is given by*

$$m_1(x) = \frac{|x_1| + \ldots + |x_N|}{|x|}$$

for $0 < |x| \leq \delta < 1$. *We have*

$$\lim_{j\to\infty} m_A(\gamma_+^j x) = \lim_{j\to\infty} \frac{\gamma_+^{-2j} a + |x_1||x_2| + \ldots + |x_{N-1}||x_N|}{\gamma_+^{-2j} b + |x|^2} = \frac{|x_1||x_2| + \ldots + |x_{N-1}||x_N|}{|x|^2} = m_+(x)$$

and

$$\lim_{j\to\infty} m_A(\gamma_-^{-j} x) = \frac{|x_1| + \ldots + |x_N|}{|x|} = m_-(x).$$

Both limits are uniform. Since m_- *and* m_+ *are bounded,* Λ_- *and* Λ_+ *are positive and finite. We have*

$$\Lambda_m = \inf_{D^{1,2}(\mathbb{R}^N)-\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\int_{\mathbb{R}^N} \frac{m_A(x)}{|x|^2} u^2\, dx} \leq \frac{1}{A} \inf_{D^{1,2}(\mathbb{R}^N)-\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\int_{1\leq|x|\leq 2} \frac{m_2(x)}{|x|^2} u^2\, dx} < \min(\Lambda_-, \Lambda_+)$$

for A *large. By Theorem 2.2,* Λ_m *with* $m = m_A$ *has a minimizer.*

Example 2.6 *Consider a sequence of functions of the form* $m_k(x) = BM_k(x) + Af(x)$, $k = 1, 2, \ldots$, *where* $A > 0$, $B > 0$ *are constants and* M_k *and* f *are continuous functions satisfying the following conditions:*

(a) $M_k(0) = 1$, $M_k(x) > 0$ *on* $\mathbb{R}^N$, $M_k(\infty) = 0$ *for* $k = 1, 2, \ldots$,

(b) $M_k(x) = k$ *on* $1 < |x| < 2$ *for* $k = 1, 2, \ldots$,

(c) $f(x) \geq 0$ *on* $\mathbb{R}^N$, $f(0) = 0$ *and* $f(\infty) = 1$.

Then $m_k(0) = B$ *and* $m_k(\infty) = A$ *for* $k = 1, 2, \ldots$. *We show that for* k *sufficiently large* m_k *satisfies the conditions of Theorem 2.3. Let* $u(x) = \exp(-|x|)$ *(one can take any other function from* $D^{1,2}(\mathbb{R}^N)$ *which is* $\not\equiv 0$ *on* $(1 < |x| < 2)$*). Thus*

$$\Lambda_{m_k} \leq \frac{\int_{\mathbb{R}^N} |\nabla(\exp(-|x|))|^2\, dx}{\int_{\mathbb{R}^N} \frac{BM_k(x)+Af(x)}{|x|^2} \exp(-2|x|)\, dx} \leq \frac{\int_{\mathbb{R}^N} |\nabla(\exp(-|x|))|^2\, dx}{B\int_{\mathbb{R}^N} \frac{M_k(x)}{|x|^2} \exp(-2|x|)\, dx} \to 0,$$

as $k \to \infty$. *So we can find* $k_\circ \geq 1$ *so that*

$$\Lambda_{m_k} < \Lambda_N \min\left(\frac{1}{A}, \frac{1}{B}\right) \quad \text{for} \quad k \geq k_\circ.$$

In Proposition 2.7, below, we described a class of weight functions m satisfying conditions of Theorem 2.3.

Proposition 2.7 *Let $m \in C(\mathbb{R}^N)$. Suppose that $m(x) \geq 0$, $m(0) > 0$ and $m(\infty) > 0$. Assume that there exists a ball $B(x_M, r)$ such that $m(x) \geq m(x_M) > 0$ for $x \in B(x_M, r)$ and $0 \notin \overline{B(x_M, r)}$. If*

$$\frac{m(0)}{m(x_M)}, \frac{m(\infty)}{m(x_M)} < \frac{r^2(N-2)^2}{2(r+|x_M|)^2(N+1)(N+2)}. \tag{2.13}$$

Then $\Lambda_m < \Lambda_N \min\left(\frac{1}{m(0)}, \frac{1}{m(\infty)}\right)$. (Hence, there exists a minimizer for Λ_m.)

Proof Let $u \in H^1_\circ(B(x_M, r)) - \{0\}$. Then

$$\int_{B(x_M,r)} \frac{m(x)}{|x|^2} u^2\, dx \geq m(x_M) \int_{B(x_M)} \frac{u^2}{|x|^2}\, dx \geq \frac{m(x_M)}{(r+|x_M|)^2} \int_{B(x_M,r)} u^2\, dx.$$

Hence

$$\frac{\int_{B(x_M,r)} |\nabla u|^2\, dx}{\int_{B(x_M,r)} \frac{m(x)}{|x|^2}\, dx} \leq \frac{(r+|x_M|)^2 \int_{B(x_M,r)} |\nabla u|^2\, dx}{m(x_M) \int_{B(x_M,r)} u^2\, dx}.$$

Since $H^1_\circ(B(x_M, r)) - \{0\} \subset \{u \in D^{1,2}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} u^2\, dx > 0\}$ we deduce from the above inequality that

$$\Lambda_m \leq \frac{(r+|x_M|)^2}{m(x_M)} \lambda_1^D(B(x_M, r)), \tag{2.14}$$

where $\lambda_1^D(B(x_M, r))$ denotes the first eigenvalue for "$-\Delta$" in $B(x_M, r)$ with the Dirichlet boundary conditions. We now estimate $\lambda_1^D = \lambda_1^D(B(x_M, r))$. We test λ_1^D with $v(x) = r - |x - x_M|$ for $x \in B(x_M, r)$. We have

$$\int_{B(x_M,r)} v^2\, dx = \int_{B(0,r)} (r-|x|)^2\, dx = \omega_N \int_0^r (r-s)^2 s^{N-1}\, ds = \frac{2\omega_N r^{N+2}}{N(N+1)(N+2)}$$

and

$$\int_{B(x_M,r)} |\nabla v|^2\, dx = \frac{\omega_N r^N}{N}.$$

Hence

$$\lambda_1^D \leq \frac{\int_{B(x_M,r)} |\nabla v|^2\, dx}{\int_{B(x_M,r)} v^2\, dx} = \frac{(N+1)(N+2)}{2r^2}.$$

Combining this with (2.14) we derive

$$\Lambda_m \leq \frac{(N+1)(N+2)(r+|x_M|)^2}{2r^2 m(x_M)}.$$

Therefore $\Lambda_m < \Lambda_N \min\left(\frac{1}{m(0)}, \frac{1}{m(\infty)}\right)$ if (2.13) holds. $\square$.

The estimate (2.13) has terms that are easy to compute, but are of course not optimal. In particular, the factor $\frac{(N+1)(N+2)}{2}$ can be replaced by the first eigenvalue of the Laplacian on a unit ball with Dirichlet boundary conditions.

If $m(x)$ is a continuous bounded and nonnegative function such that $m(x) \leq m(0)$ on $\mathbb{R}^N$ and $m(0) > 0$ (or $m(x) \leq m(\infty)$ on $\mathbb{R}^N$, $m(\infty) > 0$), then Λ_m does not have a minimizer. Indeed, suppose that $m(x) \leq m(0)$ on $\mathbb{R}^N$ and that Λ_m has a minimizer u. Then by the Hardy inequality we obtain

$$\frac{\Lambda_N}{m(0)} \geq \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} u^2\,dx} \geq \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{m(0)\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx} \geq \frac{\Lambda_N}{m(0)}.$$

So u is a minimizer for Λ_N, which is impossible.

We now construct a ground state with infinite $D^{1,2}$ norm.

Theorem 2.8 *Let $\gamma > 1$ and assume that the function $m \in L^\infty(\mathbb{R}^N)$ satisfies*

$$m(\gamma x) = m(x) \quad \text{for} \quad x \in \mathbb{R}^N. \tag{2.15}$$

Then the form Q_V with $V(x) = \frac{m(x)}{|x|^2}$ and $\Lambda_V = \Lambda_\circ$ (see (2.17) below) admits a ground state v satisfying

$$v(\gamma x) = \gamma^{\frac{2-N}{2}} v(x) \quad \text{for} \quad x \in \mathbb{R}^N. \tag{2.16}$$

The function v is uniquely defined by its values on $A_\gamma = \{x \in \mathbb{R}^N;\ 1 < |x| < \gamma\}$ and moreover the function $v_{|A_\gamma}$ is a minimizer for the problem

$$\Lambda_\circ = \inf\left\{ \frac{\int_{A_\gamma} |\nabla v|^2\,dx}{\int_{A_\gamma} \frac{m(x)}{|x|^2} u^2\,dx};\ u \in H^1(A_\gamma) - \{0\},\ u(\gamma x) = \gamma^{\frac{2-N}{2}} u(x) \ \text{ for } |x| = 1 \right\}. \tag{2.17}$$

Proof The problem (2.17) is a compact variational problem that has a minimizer v which satisfies the equation

$$-\Delta v = \Lambda_\circ \frac{m(x)}{|x|^2} v, \quad x \in A_\gamma,$$

with the Neumann boundary conditions. Since the test functions satisfy $u(\gamma x) = \gamma^{\frac{2-N}{2}} u(x)$ for $|x| = 1$, one has

$$\frac{\partial v}{\partial r}(\gamma x) = \gamma^{-\frac{N}{2}} \frac{\partial v}{\partial r}(x) \quad \text{for} \quad |x| = 1. \tag{2.18}$$

Note that $|v|$ is also a minimizer, so we may assume that v is nonnegative. We now extend the function v from A_γ to $\mathbb{R}^N - \{0\}$ by using (2.16) and denote the extended function again by v. Since v satisfies (2.17), the extended function v is of class $C^1(\mathbb{R}^N - \{0\})$ and satisfies the equation

$$-\Delta v = \Lambda_\circ \frac{m(x)}{|x|^2} v$$

in a weak sense. From this and the Harnack inequality on bounded subsets of $\mathbb{R}^N - \{0\}$ it follows that v is positive on $\mathbb{R}^N - \{0\}$ and subsequently there exists a constant $C > 0$ such that

$$C^{-1}|x|^{\frac{2-N}{2}} \le v(x) \le C|x|^{\frac{2-N}{2}}. \tag{2.19}$$

We can now explain the choice of the exponent $\frac{2-N}{2}$ in the constraint $u(\gamma x) = \gamma^{\frac{2-N}{2}}u(x)$ from (2.17): with any other choice the resulting Neumann condition would not yield the continuity of the derivatives of the extended function v on the spheres $|x| = \gamma^j$, $j \in \mathbb{N}$. Finally, we show that v is a ground state for the corresponding quadratic form Q with $V(x) = \Lambda_\circ \frac{m(x)}{|x|^2}$. Using the ground state formula (2.7) from [28] and (2.19), we have with $w_k(x) = |x|^{\frac{1}{k}}$ for $|x| \le 1$ and $w_k(x) = |x|^{-\frac{1}{k}}$ for $|x| \ge 1$,

$$\begin{aligned} Q(vw_k) &= \int_{\mathbb{R}^N} v^2|\nabla w_k|^2\,dx \le C\int_{\mathbb{R}^N} |x|^{2-N}|\nabla w_k|^2\,dx \\ &\le \frac{C}{k^2}\int_0^1 r^{-1+\frac{2}{k}}\,dr + \frac{C}{k^2}\int_1^\infty r^{-1-\frac{2}{k}}\,dr \le \frac{C}{k} \to 0 \end{aligned}$$

as $k \to \infty$. Since $vw_k \to v$ uniformly on compact sets, this implies that v is a ground state for Q. By (2.19) and the Sobolev inequality, $v \notin D^{1,2}(\mathbb{R}^N)$. $\square$

3 Perturbations from virtual ground states

In this section we show that if a potential term admits a (generalized or *large or* virtual) ground state, then its weakly continuous perturbations in the suitable direction will admit a ground state with the finite $D^{1,2}$ norm. Then we investigate potentials that do not give rise to a ground state with finite $D^{1,2}$ norm.

We need the following existence result.

Proposition 3.1 *Let $V_\circ \in L^{\frac{N}{2},\infty}(\mathbb{R}^N)$ be positive on a set of positive measure and let*

$$\Lambda_\circ = \inf_{u\in D^{1,2}(\mathbb{R}^N),\, \int_{\mathbb{R}^N} V_\circ u^2\,dx=1} \int_{\mathbb{R}^N} |\nabla u|^2\,dx. \tag{3.1}$$

Assume that $V_1 \in L^{\frac{N}{2},\infty}(\mathbb{R}^N)$ is positive on a set of positive measure and that the functional $\int_{\mathbb{R}^N}\big(V_1(x) - V_\circ(x)\big)u^2\,dx$ is weakly continuous in $D^{1,2}(\mathbb{R}^N)$ and let

$$\Lambda_1 = \inf_{u\in D^{1,2}(\mathbb{R}^N),\, \int_{\mathbb{R}^N} V_1 u^2\,dx=1} \int_{\mathbb{R}^N} |\nabla u|^2\,dx. \tag{3.2}$$

If $\Lambda_1 < \Lambda_\circ$, then there exists a minimizer for Λ_1.

Proof Let $\{u_k\} \subset D^{1,2}(\mathbb{R}^N)$ be a minimizing sequence for (3.2), that is, $\int_{\mathbb{R}^N} V_1(x)u_k^2\,dx = 1$ and $\int_{\mathbb{R}^N} |\nabla u_k|^2\,dx \to \Lambda_1$. We may assume that, up to a subsequence, $u_k \rightharpoonup w$ in $D^{1,2}(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N, V_1(x)\,dx)$. Let $v_k = u_k - w$. Then

$$\begin{aligned} 1 &= \int_{\mathbb{R}^N} V_1(x)u_k^2\,dx = \int_{\mathbb{R}^N} V_1(x)v_k^2\,dx + \int_{\mathbb{R}^N} V_1(x)w^2\,dx + o(1) = \int_{\mathbb{R}^N} V_1(x)w^2\,dx \\ &+ \int_{\mathbb{R}^N} (V_1(x) - V_\circ(x))v_k^2\,dx + \int_{\mathbb{R}^N} V_\circ(x)v_k^2\,dx + o(1) \\ &= \int_{\mathbb{R}^N} V_\circ(x)v_k^2\,dx + \int_{\mathbb{R}^N} V_1(x)w^2\,dx + o(1). \end{aligned}$$

Let $t = \int_{\mathbb{R}^N} V_1(x)w^2\,dx$. Then $\int_{\mathbb{R}^N} V_\circ(x)v_k^2\,dx \to 1 - t$. Assuming that $t < 1$ we get

$$\Lambda_1 = \int_{\mathbb{R}^N} |\nabla v_k|^2\,dx + \int_{\mathbb{R}^N} |\nabla w|^2\,dx + o(1) \geq \Lambda_\circ(1-t) + \Lambda_1 t + o(1).$$

From this we deduce that $\Lambda_1 \geq \Lambda_\circ$ which is impossible. Hence $\int_{\mathbb{R}^N} V_1(x)w^2\,dx = 1$. From this and the lower semi-continuity of the norm with respect to weak convergence, we derive that w is a minimizer and $u_k \to w$ in $D^{1,2}(\mathbb{R}^N)$.

Proposition 3.1 is related to Theorem 1.7 in [32] which asserts that a potential of the form $V(x) = \frac{1}{|x|^2} + g(x)$, with a subcritical potential g (for the definition of a subcritical potential see [32]) has a principal eigenfunction. This follows from the fact that g is weakly continuous in $D^{1,2}(\mathbb{R}^N)$ (see [30]) and the potential g admits a principal eigenfunction.

Remark 3.2 **(i)** *If $V_1 > V_\circ$, then $\Lambda_1 \leq \Lambda_\circ$, but not necessarily $\Lambda_1 < \Lambda_\circ$*

(ii) *If in Proposition 3.1 assumption $\Lambda_1 < \Lambda_\circ$ is replaced by $\Lambda_\circ < \Lambda_1$, then $\Lambda_\circ$ is attained.*

Example 3.3 *Let M be a continuous function $\mathbb{R}^N$ such that $M \geq 0$, $\not\equiv 0$ on $\mathbb{R}^N$ and $M(0) = M(\infty) = 0$. Define $m_{A,B}(x) = BM(x) + A$, where $A > 0$ and $B > 0$ are constants. Let $V_1(x) = \frac{m_{A,B}(x)}{|x|^2}$ and $V_\circ(x) = \frac{A}{|x|^2}$. The functional $\int_{\mathbb{R}^N}(V_1(x) - V_\circ(x))u^2\,dx = \int_{\mathbb{R}^N} \frac{BM(x)}{|x|^2}u^2\,dx$ is weakly continuous in $D^{1,2}(\mathbb{R}^N)$. It is easy to show that for every $A > 0$ there exists $B_\circ > 0$ such that $\Lambda_1 < \Lambda_\circ$ for $B > B_\circ$. By Proposition 3.1 Λ_1 has a minimizer for $B > B_\circ$.*

We now give a sufficient condition for the inequality $\Lambda_1 < \Lambda_\circ$.

Theorem 3.4 *Suppose that V_1 and $V_\circ$ satisfy assumptions of Proposition 3.1. Moreover, assume that the quadratic form $Q_{V_\circ}$ has a positive ground state v, possibly with infinite $D^{1,2}$ norm, and that, if $\{v_k\} \subset C_\circ^\infty(\mathbb{R}^N)$ is a null sequence corresponding to $\Lambda_\circ$, then*

$$\limsup_{k\to\infty} \int_{\mathbb{R}^N} (V_1(x) - V_\circ(x))v_k^2\,dx > 0.$$

Then $\Lambda_1 < \Lambda_\circ$ and Λ_1 has a minimizer.

Proof It suffices to show that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \Lambda_\circ \int_{\mathbb{R}^N} V_1(x) u^2 \, dx \geq 0$$

fails for some $u \in D^{1,2}(\mathbb{R}^N)$. We have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v_k|^2 \, dx \quad &- \quad \Lambda_\circ \int_{\mathbb{R}^N} V_1(x) v_k^2 \, dx = Q_{V_\circ}(v_k) - \Lambda_\circ \int_{\mathbb{R}^N} (V_1(x) - V_\circ(x)) v_k^2 \, dx \\ &= \quad o(1) - \Lambda_\circ \int_{\mathbb{R}^N} (V_1(x) - V_\circ(x)) v_k^2 \, dx < 0\end{aligned}$$

for sufficiently large k, which completes the proof of the theorem. □

Note that the conditions of Theorem 3.4 are satisfied if, in particular, $V_1 \geq V_\circ$ on $\mathbb{R}^N$, with the strict inequality on a set of positive measure. Indeed, the sequence $\{v_k\}$ converges weakly in $H^1_{\text{loc}}(\mathbb{R}^N)$ to $v > 0$ and the condition $\limsup_{k\to\infty} \int_{\mathbb{R}^N} (V_1(x) - V_\circ(x)) v_k^2 \, dx > 0$ follows from the Fatou lemma.

The situation becomes different if $Q_{V_\circ}$ does not have a ground state. The absence of the ground state is stable property under small (in some sense) compact perturbation, but not under compact perturbations that are not small.

Theorem 3.5 *Assume that $V_\circ$ satisfies the conditions of Proposition 3.1 and that (1.5) holds (this occurs under conditions of Theorem 1.4 if $Q_{V_\circ}$ has no ground state). Let W be as in (1.5). Then for every $t \in \left(0, \frac{1}{\Lambda_\circ}\right)$ the functional $Q_{V_\circ + tW}$ has no ground state and $\Lambda_{V_\circ + tW} = \Lambda_{V_\circ}$. Furthermore, if the functional $\int_{\mathbb{R}^N} W(x) u^2 \, dx$ is weakly continuous in $D^{1,2}(\mathbb{R}^N)$, the same conclusion holds for $-\infty < t < 0$.*

Proof First we observe that the constants $\Lambda_\circ$ and Λ_1 corresponding to $V_\circ$ and $V_1 = V_\circ + tW$, respectively, are equal. Indeed, since $V_1 > V_\circ$, one has $\Lambda_1 \leq \Lambda_\circ$ by monotonicity. On the other hand, it follows from (1.5) that

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \Lambda_\circ \int_{\mathbb{R}^N} (V_\circ(x) + tW(x)) u^2 \, dx \geq 0$$

for $t \in \left(0, \frac{1}{\Lambda_\circ}\right)$ which implies $\Lambda_1 \geq \Lambda_\circ$. Let $v_k \in C_\circ^\infty(\mathbb{R}^N - Z)$ satisfy $Q_{V_1}(v_k) \to 0$. Then

$$(1 - \Lambda_\circ t) \int_{\mathbb{R}^N} W v_k^2 \, dx \leq Q_{V_1}(v_k) \to 0,$$

which implies that,up to subsequence, $v_k \to 0$ a.e. If v_k were a null sequence, it would converge in $H^1_{\text{loc}}(\mathbb{R}^N)$ and it would have a limit zero. Therefore Q_{V_1} admits no null sequence and consequently no ground state. Assume now that the functional $\int_{\mathbb{R}^N} W(x) u^2 \, dx$ is weakly continuous in $D^{1,2}(\mathbb{R}^N)$. Let $\{w_k\} \subset D^{1,2}(\mathbb{R}^N)$ be a minimizing sequence for $\Lambda_\circ$. If $\{w_k\}$ has a subsequence weakly convergent in $D^{1,2}(\mathbb{R}^N)$ to some $w \neq 0$, then it is easy to see that $|w|$

would be a minimizer for $\Lambda_\circ$ and thus a ground state for $Q_{\Lambda_\circ}$. Therefore $w_k \rightharpoonup 0$. By the weak continuity of $\int_{\mathbb{R}^N} W(x)u^2\,dx$ we get

$$\int_{\mathbb{R}^N} V_1(x)w_k^2\,dx = \int_{\mathbb{R}^N} V_\circ(x)w_k^2\,dx + o(1) = 1 + o(1)$$

and thus

$$\Lambda_1 \le \int_{\mathbb{R}^N} |\nabla w_k|^2\,dx = \Lambda_\circ + o(1).$$

This yields $\Lambda_1 \le \Lambda_\circ$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \quad - \quad & \Lambda_1 \int_{\mathbb{R}^N} V_1(x)u^2\,dx \\ \ge \quad & \frac{\Lambda_1}{\Lambda_\circ}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \Lambda_\circ \int_{\mathbb{R}^N} V_1(x)u^2\,dx\right. \\ = \quad & \frac{\Lambda_1}{\Lambda_\circ}\left(Q_{V_\circ}(u) - t\Lambda_\circ \int_{\mathbb{R}^N} W(x)u^2\,dx\right) \ge \Lambda_1 \int_{\mathbb{R}^N} \left(\Lambda_\circ^{-1} - t\right)W(x)u^2\,dx.\end{aligned}$$

Since $t < 0$, this implies that Q_{V_1} has no ground state. $\square$

Theorem 3.5 concerns with small perturbations of a potential that does not change the constant Λ or the absence of a ground state. The next theorem shows that a compact perturbation of the potential term yields a ground state of finite $D^{1,2}(\mathbb{R}^N)$ norm.

Theorem 3.6 *Assume that $V_\circ$ satisfies conditions of Proposition 3.1 and that $W \in L^{2,\infty}(\mathbb{R}^N)$ is such that the functional $\int_{\mathbb{R}^N} W(x)u^2\,dx$ is weakly continuous in $D^{1,2}(\mathbb{R}^N)$. Then for every $\lambda \in \big(0, \Lambda_\circ\big)$ there exists $\sigma \in \mathbb{R}$ such that $Q_{V_\circ+\sigma W}$ has a ground state of finite $D^{1,2}(\mathbb{R}^N)$ norm corresponding to the energy constant (3.2).*

Proof Assume without loss of generality that W is positive on a set of positive measure. Let $0 < \lambda < \Lambda_\circ$ and consider

$$\sigma = \inf_{u\in D^{1,2}(\mathbb{R}^N),\, \int_{\mathbb{R}^N} W(x)u^2\,dx=1} \lambda^{-1}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \lambda \int_{\mathbb{R}^N} V_\circ(x)u^2\,dx\right).$$

Since $\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \lambda \int_{\mathbb{R}^N} V_\circ(x)u^2\,dx\right)^{\frac{1}{2}}$ defines an equivalent norm on $D^{1,2}(\mathbb{R}^N)$ it is easy to show that there exists a minimizer for σ. It is clear that this minimizer is also a ground state of $Q_{V_\circ+\sigma W}$ corresponding to the optimal constant λ. $\square$

If we assume additionally that W is positive on a set of positive measure, then it is easy to show that σ is a continuous decreasing function of λ with $\lim_{\lambda\to 0}\sigma(\lambda) = +\infty$ and $\sigma_\circ = \lim_{\lambda\to\Lambda_\circ}\sigma(\lambda) \ge 0$. In particular, if (1.5) holds with a weight $W_\circ$ satisfying $W_\circ \ge \alpha W$, then $\sigma_\circ \ge \alpha$. In other words, given $V_\circ$ and W as in Theorem 3.6, the potential $V_\circ + \sigma W$ admits a ground state whenever $\sigma \ge \sigma_\circ$.

For further results of that nature we refer to paper [32].

4 Behaviour of a ground state around 0

In what follows we consider the potential of the Hardy type $V(x) = \frac{m(x)}{|x|^2}$, where $m(x)$ is continuous and $m(0) > 0$ and $m(\infty) > 0$. The corresponding ground state, if it exists, is denoted by ϕ_1, which is chosen to be positive on $\mathbb{R}^N$. Obviously the ground state ϕ satisfies the equation

$$\Delta u = \Lambda_m \frac{m(x)}{|x|^2} u \quad \text{in } \mathbb{R}^N \tag{4.1}$$

in a weak sense.

We need the following extension of the Hardy inequality: let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $0 \in \bar{\Omega}$, then for every $\delta > 0$ there exists a constant $A(\delta, \Omega) > 0$ such that

$$\int_\Omega \frac{u^2}{|x|^2}\, dx \leq \big(\frac{1}{\Lambda_N} + \delta\big) \int_\Omega |\nabla u|^2\, dx + A(\delta, \Omega) \int_\Omega u^2\, dx \tag{4.2}$$

for every $u \in H^1(\Omega)$ (see [11]).

Proposition 4.1 *Let*

$$\Lambda_m < \Lambda_N \min\left(\frac{1}{m(0)}, \frac{1}{m(\infty)}\right).$$

Then $\phi_1 \in L^{2^*(1+\delta)}\big(B(0,r)\big)$ for some $\delta > 0$ and $r > 0$.

Proof Let $\Phi \in C^1(\mathbb{R}^N)$ be such that $\Phi(x) = 1$ on $B(0,r)$, $\Phi(x) = 0$ on $\mathbb{R}^N - B(0,2r)$, $0 \leq \Phi(x) \leq 1$ on $\mathbb{R}^N$ and $|\nabla\Phi(x)| \leq \frac{2}{r}$. For simplicity we set $\lambda = \Lambda_m$, $u = \phi_1$. We define $v = \Phi^2 u \min(u, L)^{p-2} = \Phi^2 u u_L^{p-2}$, where $L > 0$ and $p > 2$. Testing (4.1) with v, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \big(|\nabla u|^2 u_L^{p-2}\Phi^2 \quad &+ \quad (p-2)\nabla u \nabla u_L u_L^{p-2}\Phi^2 + 2\nabla u \nabla\Phi u u_L^{p-2}\Phi\big)\, dx \\ &= \quad \lambda \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} u^2 u_L^{p-2}\Phi^2\, dx.\end{aligned}$$

Applying the Young inequality to the third term on the left side, we get

$$\begin{aligned}(1-\eta)\int_{\mathbb{R}^N} |\nabla u|^2 u_L^{p-2}\Phi^2\, dx \quad &+ \quad (p-2)\int_{\mathbb{R}^N} \nabla u \nabla u_L u_L^{p-2}\Phi^2\, dx \\ &\leq \quad \lambda \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} u^2 u_L^{p-2}\Phi^2\, dx + C(\eta)\int_{\mathbb{R}^N} u^2 u_L^{p-2}|\nabla\Phi|^2\, dx,\end{aligned}$$

where $\eta > 0$ is a small number to be suitably chosen. Since the second integral on the left side is nonnegative, this inequality can be rewritten in the following form

$$\begin{aligned}(1-\eta)\int_{\mathbb{R}^N} |\nabla u|^2 u_L^{p-2}\Phi^2\, dx \quad &+ \quad (1-\eta)(p-2)\int_{\mathbb{R}^N} \nabla u \nabla u_L u_L^{p-2}\Phi^2\, dx \\ &\leq \quad \lambda \int_{\mathbb{R}^N} \frac{m(x)}{|x|^2} u^2 u_L^{p-2}\Phi^2\, dx + C(\eta)\int_{\mathbb{R}^N} u^2 u_L^{p-2}|\nabla\Phi|^2\, dx.\end{aligned}$$

Multiplying this inequality by $\frac{p+2}{4}$ and noting that $\frac{p+2}{4} > 1$, we get

$$(4.3)\qquad (1-\eta)\left[\int_{\mathbb{R}^N} |\nabla u|^2 u_L^{p-2}\Phi^2\,dx + \frac{p^2-4}{4}\int_{\mathbb{R}^N} \nabla u \nabla u_L u_L^{p-2}\Phi^2\,dx\right] \le \frac{\lambda(p+2)}{4}\int_{\mathbb{R}^N}\frac{m(x)}{|x|^2}u^2u_L^{p-2}\Phi^2\,dx + \frac{C(\eta)(p+2)}{4}\int_{\mathbb{R}^N} u^2u_L^{p-2}|\nabla\Phi|^2\,dx.$$

We now observe that

$$\int_{\mathbb{R}^N} |\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\Phi^2\,dx = \int_{\mathbb{R}^N}|\nabla u|^2u_L^{p-2}\Phi^2\,dx + \frac{p^2-4}{4}\int_{\mathbb{R}^N}|\nabla u_L|^2u_L^{p-2}\Phi^2\,dx.$$

Hence (4.3) takes the form

$$(4.4)\qquad (1-\eta)\int_{\mathbb{R}^N}|\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\Phi^2\,dx \le \frac{\lambda(p+2)}{4}\int_{\mathbb{R}^N}\frac{m(x)}{|x|^2}u^2u_L^{p-2}\Phi^2\,dx + \frac{C(\eta)(p+2)}{4}\int_{\mathbb{R}^N}u^2u_L^{p-2}|\nabla\Phi|^2\,dx.$$

Since $\frac{\lambda m(0)}{\Lambda_N} < 1$, we can choose $\epsilon_1 > 0$ so that $\frac{\lambda}{\Lambda_N}(m(0)+\epsilon_1) < 1$. By the continuity of m there exists $0 < r_1 < r$ such that $m(x) \le m(0)+\epsilon_1$ for $x \in B(0,r_1)$. This is now used to estimate the first integral on the right side of (4.4):

$$\frac{\lambda(p+2)}{4}\int_{\mathbb{R}^N}\frac{m(x)}{|x|^2}u^2u_L^{p-2}\Phi^2\,dx \le \frac{\lambda(p+2)}{4}\int_{B(0,r_1)}\frac{m(0)+\epsilon_1}{|x|^2}u^2u_L^{p-2}\,dx + \frac{\lambda(p+2)\|m\|_\infty}{4r_1^2}\int_{B(0,2r)}u^2u_L^{p-2}\,dx.$$

Applying the Hardy inequality (4.2), we get

$$\frac{\lambda(p+2)}{4}\int_{B(0,r)}\frac{m(x)}{|x|^2}u^2u_L^{p-2}\,dx \le \frac{\lambda(p+2)}{4}(m(0)+\epsilon_1)\big(\frac{1}{\Lambda_N}+\epsilon\big)\int_{B(0,r_1)}|\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx + \left(\frac{\lambda(p+2)}{4}A(B(0,r_1),\epsilon) + \frac{\lambda(p+2)\|m\|_\infty}{4r_1^2}\right)\int_{B(0,2r)}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx$$

for every $\epsilon > 0$. Inserting this estimate into (4.4) we obtain
(4.5)

$$\left(1-\eta-\frac{\lambda(p+2)}{4}(m(0)+\epsilon_1)\big(\frac{1}{\Lambda_N}+\epsilon\big)\right)\int_{B(0,r)}|\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx \le C_1\int_{B(0,2r)}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx,$$

where $C_1 = \frac{\lambda(p+2)}{4}A(B(0,r_1),\epsilon) + \frac{\lambda(p+2)\|m\|_\infty}{4r_1^2} + \frac{(p+2)C(\eta)}{r^2}$. We put $p = 2+\delta$, $\delta > 0$. We now observe that we can choose δ and ϵ so small that

$$\lambda\big(1+\frac{\delta}{4}\big)(m(0)+\epsilon_1)\big(\frac{1}{\Lambda_N}+\epsilon\big) = \frac{\lambda}{\Lambda_N}\big(1+\frac{\delta}{4}\big)(m(0)+\epsilon_1) + \lambda\epsilon\big(1+\frac{\delta}{4}\big)(m(0)+\epsilon_1) < 1.$$

We point out that we have used here the inequality $\frac{\lambda}{\Lambda_N}(m(0)+\epsilon_1)<1$. With this choice of ϵ and δ we now choose $\eta>0$ so small that

$$C_2 := 1-\eta-\lambda\big(1+\frac{\delta}{4}\big)(m(0)+\epsilon_1)\big(\frac{1}{\Lambda_N}+\epsilon\big)>0.$$

Finally, we apply the Sobolev inequality in $H^1(B(0,r))$ and deduce

$$SC_2\bigg(\int_{B(0,r)}|uu_L^{\frac{p}{2}-1}|^{2^*}\,dx\bigg)^{\frac{2}{2^*}} \le (C_1+C_2)\int_{B(0,2r)}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx,$$

where S denotes the best Sobolev constant of the embedding of $H^1(B(0,r))$ into $L^{2^*}(B(0,r))$. Letting $L\to\infty$ we deduce that $u\in L^{2^*(1+\frac{\delta}{2})}(B(0,r))$. So the assertion holds with $\delta_\circ=\frac{\delta}{2}$. □

We now establish the higher integrability property of the principal eigenfunction on $\mathbb{R}^N\setminus B(0,R)$. Although this will not be used in the sequel, we add it for the sake of completeness. We denote by $D^{1,2}(\mathbb{R}^N\setminus B(0,R))$ the Sobolev space defined by

$$D^{1,2}(\mathbb{R}^N\setminus B(0,R))=\{u:\ \nabla u\in L^2(\mathbb{R}^N\setminus B(0,R))\ \text{ and }\ u\in L^{2^*}(\mathbb{R}^N\setminus B(0,R))\}.$$

Lemma 4.2 *For every $\delta>0$, there exists a constant $A=A(\delta,R)>0$ such that*

$$\int_{|x|\ge R}\frac{u^2}{|x|^2}\,dx\le\big(\frac{1}{\Lambda_N}+\delta\big)\int_{|x|\ge R}|\nabla u|^2\,dx+A\int_{R\le|x|\le R+1}u^2\,dx$$

for every $u\in D^{1,2}(\mathbb{R}^N\setminus B(0,R))$.

Proof Let $\Phi\in C^1(\mathbb{R}^N)$ be such that $\Phi(x)=0$ on $\overline{B(0,R)}$, $\Phi(x)=1$ on $\mathbb{R}^N\setminus B(0,R+1)$, $0\le\Phi(x)\le 1$ on $\mathbb{R}^N\setminus B(0,R)$ and $|\nabla\Phi(x)|\le\frac{2}{R}$ on $\mathbb{R}^N$. Then $u\Phi\in D^{1,2}(\mathbb{R}^N)$ and by the Hardy and Young inequalities, we have

$$\begin{aligned}
\int_{|x|\ge R}\frac{u^2}{|x|^2}\,dx &= \int_{|x|\ge R}\frac{(u\Phi)^2}{|x|^2}\,dx+\int_{|x|\ge R}\frac{(1-\Phi^2)u^2}{|x|^2}\,dx\\
&\le \Lambda_N^{-1}\int_{|x|\ge R}|\nabla(u\Phi)|^2\,dx+\frac{1}{R^2}\int_{R\le|x|\le R+1}u^2\,dx\\
&\le \Lambda_N^{-1}\int_{|x|\ge R}|\nabla u|^2\Phi^2\,dx+\Lambda_N^{-1}\int_{|x|\ge R}u^2|\nabla\Phi|^2\,dx\\
&+ 2\Lambda_N^{-1}\int_{|x|\ge R}u\Phi\nabla u\nabla\Phi\,dx+\frac{1}{R^2}\int_{R\le|x|\le R+1}u^2\,dx\\
&\le \big(\Lambda_N^{-1}+\delta\big)\int_{|x|\ge R}|\nabla u|^2\,dx+\big(\Lambda_N^{-1}+C(\delta)\big)\int_{|x|\ge R}u^2|\nabla\Phi|^2\,dx\\
&+ \frac{1}{R^2}\int_{R\le|x|\le R+1}u^2\,dx
\end{aligned}$$

and the result follows with $A(\delta,R)=\frac{4}{R^2}\big(\Lambda_N^{-1}+C(\delta)\big)+\frac{1}{R^2}$. □

Proposition 4.3 *Suppose that $m(\infty) > 0$ and $\Lambda_m < \Lambda_N \min\left(\frac{1}{m(0)}, \frac{1}{m(\infty)}\right)$. Let ϕ_1 be the principal eigenfunction of problem (4.1). Then there exist $\delta > 0$ and $R > 0$ such that $\phi \in L^{2^*(1+\delta)}(\mathbb{R}^N \setminus B(0,R))$.*

Proof We modify the argument used in the proof of Proposition 4.1. Since $\Lambda_m < \frac{\Lambda_N}{m(\infty)}$, there exist $\epsilon > 0$ and $R > 0$ such that $\frac{\Lambda_m}{\Lambda_N}(m(\infty)+\epsilon) < 1$ and $m(x) < m(\infty)+\epsilon$ for $|x| \geq R$. Let $\Psi \in C^1(\mathbb{R}^N)$ be such that $\Psi(x) = 0$ on $B(0,R)$, $\Psi(x) = 1$ on $\mathbb{R}^N - B(0,R+1)$, $0 \leq \Psi(x) \leq 1$ on $\mathbb{R}^N$ and $|\nabla\Psi(x)| \leq \frac{2}{R}$ on $\mathbb{R}^N$. Let $\lambda = \Lambda_m$, $u = \phi_1$ and $v = uu_L^{p-2}\Psi^2$, where $L > 1$, $p > 2$ and $u_L = \min(u, L)$. It is clear that $v \in D^{1,2}(\mathbb{R}^N)$. Testing (4.1) with v and applying the Young inequality, we obtain

$$\begin{aligned}(1-\eta)\int_{\mathbb{R}^N} |\nabla u|^2 u_L^{p-2}\Psi^2\,dx \;&+\; (p-2)\int_{\mathbb{R}^N} \nabla u \nabla u_L u_L^{P-2}\Psi^2\,dx \\ &\leq \lambda\int_{\mathbb{R}^N}\frac{m(x)}{|x|^2}u^2u_L^{p-2}\Psi^2\,dx + C(\eta)\int_{\mathbb{R}^N} u^2u_L^{p-2}|\nabla\Psi||^2\,dx.\end{aligned}$$

From this, as in the proof of Proposition 4.1, we derive that

$$\begin{aligned}(1-\eta)\int_{\mathbb{R}^N}|\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\Psi^2\,dx &\leq \frac{\lambda(p+2)}{4}\int_{\mathbb{R}^N}\frac{m(x)}{|x|^2}u^2u_L^{p-2}\Psi^2\,dx \\ &+ \frac{C(\eta)(p+2)}{4}\int_{\mathbb{R}^N}u^2u_L^{p-2}|\nabla\Psi|^2\,dx.\end{aligned} \tag{4.6}$$

We now estimate the first integral on the right side of (4.6). Using Lemma 4.2 we have for every $\epsilon_1 > 0$

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{m(x)}{|x|^2}u^2u_L^{p-2}\Psi^2\,dx &\leq (m(\infty)+\epsilon)\int_{|x|\geq R+1}\frac{\big(uu_L^{\frac{p}{2}-1}\big)^2}{|x|^2}\,dx \\ &+ (m(\infty)+\epsilon)\int_{R\leq|x|\leq R+1}\frac{\big(uu_L^{\frac{p}{2}-1}\big)^2}{|x|^2}\,dx \\ &\leq \big(\Lambda_N^{-1}+\epsilon_1\big)(m(\infty)+\epsilon)\int_{|x|\geq R+1}|\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx \\ &+ A(\epsilon_1,R)(m(\infty)+\epsilon)\int_{R+1\leq|x|\leq R+2}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx \\ &+ \frac{m(\infty)+\epsilon}{R^2}\int_{R\leq|x|\leq R+1}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx.\end{aligned}$$

Inserting this into (4.6) we obtain

$$\begin{aligned}&\left[1-\eta-\frac{\lambda(p+2)}{4}\big(\Lambda_N^{-1}+\epsilon_1\big)(m(\infty)+\epsilon)\right]\int_{|x|\geq R+1}|\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx \\ &\leq C_1(\delta,\epsilon_1,R)\int_{R\leq|x|\leq R+2}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx,\end{aligned} \tag{4.7}$$

where

$$C_1(\delta,\epsilon_1,R):=\frac{\lambda(p+2)}{4}(m(\infty)+\epsilon)A(\epsilon_1,R)+\frac{\lambda(p+2)}{4R^2}(m(\infty)+\epsilon)+\frac{C(\eta)(p+2)}{R^2}.$$

We now set $p=2+\delta$. We choose $\delta>0$ and $\epsilon_1>0$ such that

$$\lambda(1+\frac{\delta}{4})\big(\Lambda_N^{-1}+\epsilon_1\big)(m(\infty)+\epsilon)<1.$$

Then we choose $\eta>0$ small enough to guarantee the inequality

$$C_2:=1-\eta-\lambda\big(1+\frac{\delta}{4}\big)\big(\Lambda_N^{-1}+\epsilon_1\big)(m(\infty)+\epsilon)>0.$$

Having chosen ϵ_1 and δ we apply the Sobolev inequality to deduce from (4.7)

$$SC_2\bigg(\int_{|x|\geq R+1}|\big(uu_L^{\frac{p}{2}-1}\big)|^{2^*}\,dx\bigg)^{\frac{2}{2^*}}\leq C_1\int_{R\leq|x|\leq R+1}\big(uu_L^{\frac{p}{2}-1}\big)^2\,dx,$$

where S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^N-B(0,R+1))$ into $L^{2^*}(\mathbb{R}^N-B(0,R+1))$. Letting $L\to\infty$, the result follows. □

Continuing with the above notations $\lambda=\Lambda_m$, $u=\phi_1$, we put $u=|x|^{-s}v$, with $s>0$ to be chosen later. We have

$$\operatorname{div}\big(|x|^{-2s}\nabla v\big)=-\lambda|x|^{-2-s}m(x)u+u\big(-s^2|x|^{-s-2}+sN|x|^{-s-2}-2s|x|^{-s-2}\big).$$

We now consider the above equation in a small ball $B(0,r)$. Since

$$\lambda=\Lambda_m<\Lambda_N\min\bigg(\frac{1}{m(0)},\frac{1}{m(\infty)}\bigg)\leq\frac{\Lambda_N}{m(0)},$$

there exists $r>0$ (small enough) such that $\lambda\max_{x\in B(0,r)}m(x)<\Lambda_N$. Let $s=\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda\bar{m}_r}$ with $\bar{m}_r=\max_{x\in B(0,r)}m(x)$, then

$$-\operatorname{div}\big(|x|^{-2s}\nabla v\big)\leq 0\quad\text{in}\quad B(0,r).\tag{4.8}$$

Let $\underline{\mathrm{m}}_r=\min_{x\in B(0,r)}m(x)$ and set $s=\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda\underline{\mathrm{m}}_r}$. Then

$$-\operatorname{div}\big(|x|^{-2s}\nabla v\big)\geq 0\quad\text{in}\quad B(0,r).\tag{4.9}$$

Proposition 4.4 *Let $m(0)>0$ and*

$$\Lambda_m<\Lambda_N\min\bigg(\frac{1}{m(0)},\frac{1}{m(\infty)}\bigg).$$

Then there exists $r>0$ such that

$$M_1|x|^{-(\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda\underline{\mathrm{m}}_r})}\leq\phi_1(x)\leq M_2|x|^{-(\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda_m\bar{m}_r})}\tag{4.10}$$

for $x\in B(0,r)$ and some constants $M_1>0$, $M_2>0$.

The lower bound follows from Proposition 2.2 in [13]. To apply it we need inequality (4.9). To establish the upper bound, we modify the argument used in paper [16]. Let η be a C^1 function such that $\eta(x) = 1$ on $B(0, r)$, $\eta(x) = 0$ on $\mathbb{R}^N \setminus B(0, \rho)$ and $|\nabla \eta(x)| \le \frac{2}{\rho - r}$ on $\mathbb{R}^N$, where $0 < r < \rho$. We use as a test function in (4.8) $w = \eta^2 v v_l^{2(t-1)} = \eta^2 v \min(v, l)^{2(t-1)}$, where $l, t > 1$. Substituting into (4.8), we obtain

$$\int_{\mathbb{R}^N} |x|^{-2s}\big(2\eta v v_l^{2(t-1)} \nabla v \nabla \eta + \eta^2 v_l^{2(t-1)} |\nabla v|^2 + 2(t-1)\eta^2 v_l^{2(t-1)} |\nabla v_l|^2\big)\, dx \le 0, \tag{4.11}$$

where $s = \sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda \bar{m}_r}$. By the Young inequality, for every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$\begin{aligned} 2\int_{\mathbb{R}^N} |x|^{-2s} \eta v v_l^{2(t-1)} \nabla \eta \nabla v \, dx &\le \epsilon \int_{\mathbb{R}^N} |x|^{-2s} \eta^2 v_l^{2(t-1)} |\nabla v|^2 \, dx \\ &+ C(\epsilon) \int_{\mathbb{R}^N} |x|^{-2s} |\nabla \eta|^2 v^2 v_l^{2(t-1)} \, dx. \end{aligned}$$

Taking $\epsilon = \frac{1}{2}$, we derive from (4.11) that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{-2s}\big(\eta^2 v_l^{2(t-1)} |\nabla v|^2 &+ 2(t-1)\eta^2 v_l^{2(t-1)} |\nabla v_l|^2\big)\, dx \\ &\le C \int_{\mathbb{R}^N} |x|^{-2s} |\nabla \eta|^2 v^2 v_l^{2(t-1)} \, dx, \end{aligned} \tag{4.12}$$

where $C > 0$ is a constant independent of l. To proceed further we use the Caffarelli - Kohn - Nirenberg inequality [9]:

$$\left(\int_{B(0,\rho)} |x|^{-bp} |w|^p \, dx\right)^{\frac{2}{p}} \le C_{a,b} \int_{B(0,\rho)} |x|^{-2a} |\nabla w|^2 \, dx \tag{4.13}$$

for every $w \in H^1_\circ\big(B(0, \rho), |x|^{-2a}\, dx\big)$, where $-\infty < a < \frac{N-2}{2}$, $a \le b \le a + 1$, $p = \frac{2N}{N-2+2(b-a)}$ and $C_{a,b} > 0$ is a constant depending on a and b. We choose

$$a = b = \sqrt{\Lambda_N} - \sqrt{\Lambda_N - \lambda \bar{m}_r} < \frac{N-2}{2}.$$

In this case we have $p = 2^*$. We then deduce from (4.12) and (4.13) with $w = \eta v v_l^{t-1}$, that

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |x|^{-2^* s} |\eta v v_l^{t-1}|^{2^*} \, dx\right)^{\frac{2}{2^*}} &\le C_{a,b} \int_{\mathbb{R}^N} |x|^{-2s} |\nabla\big(\eta v v_l^{t-1}\big)|^2 \, dx \\ &\le 2C_{a,b} \int_{\mathbb{R}^N} |x|^{-2s}\big(|\nabla \eta|^2 v^2 v_l^{2(t-1)} + \eta^2 v_l^{2(t-1)} |\nabla v|^2 \\ &+ (t-1)^2 \eta^2 v_l^{2(t-1)} |\nabla v_l|^2\big)\, dx \\ &\le Ct \int_{\mathbb{R}^N} |x|^{-2^* s} |\nabla \eta|^2 v^2 v_l^{2(t-1)} \, dx. \end{aligned} \tag{4.14}$$

We now observe that

$$\int_{\mathbb{R}^N} |x|^{-2^*s}|\eta|^{2^*} v^2 v_l^{2^*t-2}\, dx \le \int_{\mathbb{R}^N} |x|^{-2^*s}|\eta v v_l^{t-1}|^{2^*}\, dx.$$

Indeed, to show this we need to check that $v^2 v_l^{2^*t-2} \le v_l^{2^*(t-1)} v^{2^*}$ on supp η. This can be verified by considering the cases $v_l = l$ and $v_l = v$. The above inequality allows us to rewrite (4.14) as

$$\left(\int_{\mathbb{R}^N} |x|^{-2^*s}|\eta|^{2^*} v^2 v_l^{2^*t-2}\, dx\right)^{\frac{2}{2^*}} \le Ct \int_{\mathbb{R}^N} |x|^{-2^*s}|\nabla\eta|^2 v^2 v_l^{2(t-1)}\, dx.$$

Due to the properties of the function η, the above inequality becomes

$$\left(\int_{B(0,r)} |x|^{-2^*s} v^2 v_l^{2^*t-2}\, dx\right)^{\frac{2}{2^*}} \le \frac{Ct}{(\rho-r)^2}\int_{B(0,\rho)} |x|^{-2^*s} v^2 v_l^{2(t-1)}\, dx. \tag{4.15}$$

One can easily check that the resulting integral on the right side is of (4.15) is finite. We now choose $\frac{N}{N-2} < t^* < (1+\delta_\circ)\frac{N}{N-2}$, where $\delta_\circ$ is a constant from Proposition 4.1. We define the sequence $t_j = t^*\big(\frac{2^*}{2}\big)^j$, $j = 0, 1, \ldots$. Setting $t = t_j$ in (4.15), we obtain

$$\left(\int_{B(0,r)} |x|^{-2^*s} v^2 v_l^{2t_{j+1}-2}\, dx\right)^{\frac{1}{2t_{j+1}}} \le \left(\frac{Ct_j}{(\rho-r)^2}\right)^{\frac{1}{2t_j}} \left(\int_{B(0,\rho)} |x|^{-2^*s} v^2 v_l^{2t_j-2}\, dx\right)^{\frac{1}{2t_j}}.$$

We put $r_j = \rho_\circ\big(1+\rho_\circ^j\big)$, $j = 0, 1, \ldots$ with $\rho_\circ$ small. Substituting in the last inequality $\rho = r_j$, $r = r_{j+1}$, we obtain

$$\left(\int_{B(0,r_{j+1})} |x|^{-2^*s} v^2 v_l^{2t_{j+1}-2}\, dx\right)^{\frac{1}{2t_{j+1}}} \le \left(\frac{Ct_j}{(\rho_\circ-\rho_\circ^2)^2\rho_\circ^{2j}}\right)^{\frac{1}{2t_j}} \left(\int_{B(0,r_j)} |x|^{-2^*s} v^2 v_l^{2t_j-2}\, dx\right)^{\frac{1}{2t_j}}. \tag{4.16}$$

Iterating gives

$$\left(\int_{B(0,r_{j+1})} |x|^{-2^*s} v^2 v_l^{2t_{j+1}-2}\, dx\right)^{\frac{1}{2t_{j+1}}} \tag{4.17}$$

$$\le \left(\frac{C}{\rho_\circ-\rho_\circ^2}\right)^{\sum_{j=0}^\infty \frac{1}{t_j}} \rho_\circ^{-\sum_{j=0}^\infty \frac{1}{t_j}} \prod_{j=0}^\infty t_j^{\frac{1}{2t_j}} \left(\int_{B(0,r_\circ)} |x|^{-2^*s} v^2 v_l^{2t^*-2}\, dx\right)^{\frac{1}{2^*}}. \tag{4.18}$$

We now notice that infinite sums and the infinite product in the above inequality are finite. Since $2^* < 2t^* < (1+\delta_\circ)2^*$, we have

$$\int_{B(0,r_\circ)} |x|^{-2^*s} v^2 v_l^{2t^*-2}\, dx \le \int_{B(0,r_\circ)} |x|^{(2t^*-2^*)s}|u|^{2t^*}\, dx \le r_\circ^{(2t^*-2^*)s}\int_{B(0,r_\circ)} |u|^{2^*t^*}\, dx < \infty. \tag{4.19}$$

We now deduce from (4.17) and (4.19) that

$$\begin{aligned} \|v_l\|_{L^{2t_{j+1}}(B(0,\rho_\circ))} &\le \|v_l\|_{L^{2t_{j+1}}(B(0,r_{j+1}))} \\ &\le r_\circ^{\frac{2^*s}{2t_{j+1}}} \left(\int_{B(0,r_{j+1})} |x|^{-2^*s} v^2 v_l^{t_{j+1}-2}\, dx\right)^{\frac{1}{2t_{j+1}}} \le C, \end{aligned}$$

where $C > 0$ is a constant independent of l and j. Letting $t_j \to \infty$ we get $\|v_l\|_{L^\infty(B(0,\rho_\circ))} \leq C$. Finally, if $l \to \infty$ we obtain $\|v\|_{L^\infty(B(0,\rho_\circ))} \leq C$ and this completes the proof. □

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